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Abstract. We study the set $|\alpha, n) = D(\alpha)|n\rangle$ at fixed n where $D(\alpha)$ is the Glauber displacement operator for bosons.

Specifically we demonstrate a closure relation and we use it for the description of operators in a generalized P representation. We apply these results to some equations of evolution which then appear in a near-classical form.

1. Introduction

The problem of the harmonic oscillator (HO) is one of the most fundamental in quantum mechanics. Its interest lies in the fact that it describes normal modes of bosons (photons, phonons, magnons, etc...) and because its eigenvectors can be used as a basis for more complicated problems. Those eigenvectors $|n\rangle$, where the number of quanta in the state is given, constitute Dirac's basis.

The question of the HO displaced by an external macroscopic force is easily solved: the eigenvectors are of the form $|\alpha, n) = D(\alpha)|n\rangle$ with α given; $D(\alpha)$ is a displacement operator. The set of $|\alpha, n)$ (α given) is complete and is a basis.

An important step has been made by Glauber (1963a, b) when he gave a closure relation with the set $|\alpha, 0)$ where α is a complex number. Moreover, the state $|\alpha, 0)$ is an eigenvector of the annihilation operator a:

$$a|\alpha, 0) = \alpha|\alpha, 0).$$

As a consequence, it factorizes correlation functions defining quantum coherence; so, they are called coherent states. Another interesting property of the $|\alpha, 0)$ is that they minimize the product $\Delta p \,.\, \Delta q$. They behave as classically as possible. The price one must pay for that is their lack of orthogonality.

In this paper we study the states $|\alpha, n)$ and their possible usefulness for the description of operators.

In § 2 we establish some properties of these states with given n and especially we show that they are an overcomplete basis, as for coherent states. But the property of factorization is lost; so they will be referred to as semicoherent states (SCS). Next we give the representation of vectors and operators.

In § 3 we generalize the notation of a P representation particular to continuous representations.

Finally, we give some applications to the derivation of equations of evolution.

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2. Semicoherent states

2.1. *Definition and some properties*

Let us write $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ the unitary translation operator introduced by Glauber. We define the SCS of order n by:

$$|\alpha, n) = D(\alpha)|n\rangle. \tag{1.1}$$

Because of the properties of the states $|n\rangle$ and of the operators a, $a^\dagger$, we have

$$|\alpha, n+1) = D(\alpha)|n+1\rangle = D(\alpha)\frac{a^\dagger}{\sqrt{n+1}}|n\rangle = D(\alpha)\frac{a^\dagger}{\sqrt{n+1}}D^\dagger(\alpha)|\alpha, n). \tag{1.2}$$

We put

$$\begin{aligned} A^\dagger(\alpha) &= D(\alpha)a^\dagger D^\dagger(\alpha) = a^\dagger - \alpha^* \\ A(\alpha) &= D(\alpha)aD^\dagger(\alpha) = a - \alpha. \end{aligned} \tag{1.3}$$

It is easy to see from equation (1.3) and similar equations that the operators $A(\alpha)$ and $A^\dagger(\alpha)$ play the role of annihilation and creation operators:

$$\begin{aligned} A^\dagger(\alpha)|\alpha, n) &= \sqrt{n+1}|\alpha, n+1) \\ A(\alpha)|\alpha, n) &= \sqrt{n}|\alpha, n-1) \\ |\alpha, n) &= \frac{(A^\dagger(\alpha))^n}{(n!)^{1/2}}|\alpha, 0) \\ N(\alpha)|\alpha, n) &= A^\dagger(\alpha)A(\alpha)|\alpha, n) = n|\alpha, n). \end{aligned} \tag{1.4}$$

The mean value of the operator $a^\dagger a$ in the SCS $|\alpha, n)$ is equal to $n + |\alpha|^2$; it is the sum of an incoherent term and of a coherent term. This justifies the name 'semicoherent state'. Perhaps, it is worthwhile to note that

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a) = \exp(\alpha A^\dagger(\alpha) - \alpha^* A(\alpha)) \tag{1.5}$$

and

$$[A(\alpha), A^\dagger(\beta)] = 1 \qquad \text{for all } \alpha, \beta.$$

Figure 1 shows the effect of these operators.

Glauber has shown that the coherent states $|\alpha, 0)$ are the eigenvectors of the annihilation operator. What about this point of view for the SCS?

First, we recall that the vector $|x_k\rangle$ is an eigenvector of rank k of the operator X if it satisfies the relations

$$\begin{aligned} (X-x)^k|x_k\rangle &= 0 \\ (X-x)^{k-1}|x_k\rangle &\neq 0 \end{aligned}$$

x is evidently an eigenvalue of X.

Then, the SCS $|\alpha, n)$ is an eigenvector of rank $n+1$ for the annihilation operator a. This is seen from

$$D(\alpha)a^{n+1}D^\dagger(\alpha)|\alpha, n) = D(\alpha)a^{n+1}|n\rangle = 0$$

and

$$D(\alpha)a^{n+1}D^\dagger(\alpha) = (a-\alpha)^{n+1}$$

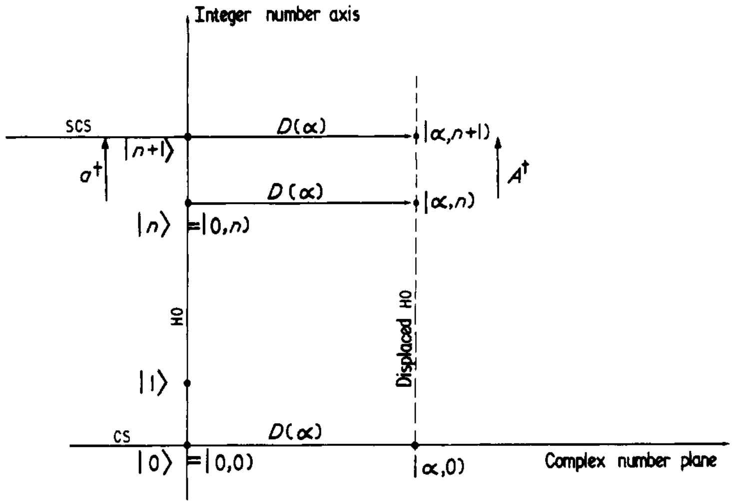

Figure 1. An illustration of the effect of operators $a^{\dagger}$, $A^{\dagger}(\alpha)$ and $D(\alpha)$. For a, $A(\alpha)$ and $D^{-1}(\alpha) = D(-\alpha)$ the arrows must be reversed.

so

$$(a-\alpha)^{n+1}|\alpha, n) = 0.$$

Similarly

$$D(\alpha)a^{n}D^{\dagger}(\alpha)|\alpha, n) = D(\alpha)a^{n}|n\rangle = (n!)^{1/2}|\alpha, 0)$$

or

$$(a-\alpha)^{n}|\alpha, n) = (n!)^{1/2}|\alpha, 0) \neq 0.$$

In the following, this definition of $|\alpha, n)$ will not be used because it is not unique: one can always add an eigenvector of rank lower than $n+1$ (except, for Glauber's case where $n = 0$).

Some scalar products will be useful in the forthcoming pages; we rapidly establish some of them. Here p and later q are positive integers:

$$\langle p|\alpha, n) = \langle p|D(\alpha)|n\rangle = \exp(\tfrac{1}{2}|\alpha|^2)\langle p|\exp(\alpha a^{\dagger})\exp(-\alpha^{*}a)|n\rangle.$$

Because of the relation

$$e^{\gamma a}|n\rangle = \sum_{i=0}^{n}\frac{(\gamma)^{i}}{i!}\sqrt{\frac{n!}{(n-i)!}}|n-i\rangle$$

and of the orthogonality $\langle p|q\rangle = \delta_{pq}$ we get:

$$\langle p|\alpha, n) = \exp(-\tfrac{1}{2}|\alpha|^2)\sum_{i=0}^{n}\sum_{j=0}^{p}(-1)^{i}\frac{\alpha^{j}\alpha^{*i}}{i!j!}\frac{\sqrt{n!p!}}{(n-i)!}\delta_{p-j,n-i}$$

which can be transformed into

$$\langle p|\alpha, n) = \exp(-\tfrac{1}{2}|\alpha|^2)\sqrt{n!p!}\,\alpha^{p-n}\sum_{i=I}^{n}(-1)^{i}\frac{|\alpha|^{2i}}{i!(n-i)!(p-n+i)!} \tag{1.6}$$

with $I = \sup\{0, n-p\}$. The SCS $|\alpha, p)$ and $|\alpha, q)$ are orthogonal:

$$(\alpha, p|\alpha, q) = \langle p|D^{\dagger}(\alpha)D(\alpha)|q\rangle = \langle p|q\rangle = \delta_{pq}. \tag{1.7}$$

2.2. Completeness of the SCS

The previous results are simple generalizations of Glauber's states and a lot of them have been used in the literature. However, a more fundamental property has not been emphasized: the set of $|\alpha, n)$ (n given) is complete (in fact, because the basis is continuous, it is overcomplete, ie it is not hilbertian).

To give a proof, we need only to demonstrate that the unit operator may be expressed as a suitable integral, over the whole complex plane α, of projection operators $|\alpha, n)(\alpha, n|$, namely:

$$1 = \frac{1}{\pi}\int \mathrm{d}^2\alpha|\alpha, n)(\alpha, n|; \tag{1.8}$$

$\mathrm{d}^2\alpha$ is the differential surface element of the complex plane.

It is sufficient to show that the equality holds for the matrix elements in a complete basis: the set of $|n\rangle$. Therefore, we have to establish that:

$$I_n(p, q) = \frac{1}{\pi}\int \mathrm{d}^2\alpha\langle p|\alpha, n)(\alpha, n|q\rangle = \delta_{pq} \qquad \text{for all } p, q. \tag{1.9}$$

Using formula (1.6) and its adjoint for the scalar products, and the relation

$$\frac{1}{\pi}\int \mathrm{d}^2\alpha \exp(-|\alpha|^2)\alpha^l(\alpha^*)^m = l!\delta_{lm} \tag{1.10}$$

equation (1.9) becomes

$$I_n(p, q) = \delta_{pq}\sum_{i,j=I}^{n}(-1)^{i+j}C_j^nC_{n-i}^pC_i^{p-n+i+j} = I_n(p, p)$$

with $I = \sup\{0, n-p\}$.

Let us calculate $I_n(p, q)$ when $p \geqslant n$, so that $I = 0$. Then

$$I_n(p, q) = \delta_{pq}\sum_{i,j=0}^{n}(-1)^{i+j}C_{n-i}^pC_j^n\frac{(p-n+i+j)!}{i!(p-n+j)!}.$$

Remarking that $(p-n+i+j)!/(p-n+j)!$ is a polynomial of degree i in j, and, using the result

$$\sum_{j=0}^{n}(-1)^jC_j^nj^m = \begin{cases} 0 & \text{for } m < n \\ (-1)^n n! & \text{for } m = n, \end{cases}$$

we obtain

$$I_n(p, q) = \delta_{pq}\sum_{i=0}^{n}(-1)^i\frac{C_{n-i}^p}{i!}(-1)^n n!\delta_{in} = \delta_{pq}. \tag{1.11}$$

This result is also valid for $p < n$ because

$$I_n(p, q) = I_n(p, p) = I_p(n, n) = I_p(n, m).$$

2.3. Representation of a vector, of an operator

Using the completeness relation, it is easy to see that, for an arbitrary state

$$|x\rangle = \frac{1}{\pi}\int d^2\alpha|\alpha, n)(\alpha, n|x\rangle.$$

In the same way, for an operator A we have:

$$A = \frac{1}{\pi^2}\int\int d^2\alpha\, d^2\beta|\alpha, m)(\alpha, m|A|\beta, n)(\beta, n|. \tag{1.12}$$

In the particular (and probably more useful) case where $m = n$, we let

$$\mathscr{A}_n(\alpha, \alpha^*, \beta, \beta^*) = \exp\{\tfrac{1}{2}(|\alpha|^2+|\beta|^2)\}(\alpha, n|A|\beta, n).$$

We do not insist on this representation because the notion of a 'P representation' seems more useful.

3. Generalized *P* representation

3.1. 'Diagonal' or P representation

Klauder (1963), using the theory of continuous representation, has demonstrated that it is possible to write a large class of operators in a diagonal form.

In our case, we give two processes for computing the $A_n^{\mathrm{D}}(\alpha^*, \alpha)$ in

$$A = \int d^2\alpha A_n^{\mathrm{D}}(\alpha^*, \alpha)|\alpha, n)(\alpha, n| \tag{2.1}$$

proving at the same time the existence of such a decomposition.

3.1.1. Functional derivation. Our treatment is not mathematically rigorous because we are concerned in fact with distributions. However, it is not really a problem because that part of mathematics is well known and leads to the results we get here.

Let $K_n = K_n(\alpha^*, \alpha, \beta^*, \beta, \gamma^*, \gamma)$ be the diagonal representation for the dyadic $|\alpha, n)(\beta, n|$. In fact K_n is a functional

$$|\alpha, n)(\beta, n| = \int d^2\gamma K_n|\gamma, n)(\gamma, n|. \tag{2.2}$$

Using the relations (2.1), (2.2) and (1.12) we obtain:

$$A_n^{\mathrm{D}}(\gamma^*, \gamma) = \pi^{-2}\int\int d^2\alpha\, d^2\beta K_n(\alpha^*, \alpha, \beta^*, \beta, \gamma^*, \gamma)A_n(\beta^*, \beta, \alpha^*, \alpha). \tag{2.3}$$

3.1.2. Differential derivation. From (1.4) and (1.8) it is easy to check that:

$$\int d^2\gamma P_n(\gamma^*, \gamma)|\gamma, n)(\gamma, n| = \pi^{-2}\int\int d^2\alpha\, d^2\beta R_0(\alpha^*, \alpha, \beta^*, \beta)|\alpha, 0)(\beta, 0| \tag{2.4}$$

with

$$R_0(\alpha^*, \alpha, \beta^*, \beta) = \pi^{-2}\int d^2\gamma P_n(\gamma^*, \gamma)\frac{(\gamma^*-\alpha^*)^n(\gamma-\beta)^n}{n!}(\alpha, 0|\gamma, 0)(\gamma, 0|\beta, 0). \tag{2.5}$$

$R_0(\alpha^*, \alpha, \beta^*, \beta)$ is the nondiagonal representation of the operator A in the coherent state basis.

If we let $\alpha = -\beta$ we get:

$$R_0(-\beta^*, -\beta, \beta^*, \beta)\exp(|\beta|^2)$$
$$= \pi^{-2}\int d^2\gamma P_n(\gamma^*, \gamma)\frac{(\gamma^*+\beta^*)^n(\gamma-\beta)^n}{n!}\exp(-|\gamma|^2)\exp(\gamma^*\beta-\beta^*\gamma). \tag{2.6}$$

In the case $n = 0$ this is Mehta's formula (Mehta 1967):

$$R_0(-\beta^*, -\beta, \beta^*, \beta)\exp(|\beta|^2) = \int d^2\gamma P_0(\gamma^*, \gamma)\exp(-|\gamma|^2)\exp(\gamma^*\beta-\beta^*\gamma). \tag{2.7}$$

We can Fourier invert the relation (2.6) to obtain:

$$P_0(\gamma^*, \gamma)\exp(-|\gamma|^2) = \{\overleftarrow{T}(+\partial_\gamma+\gamma^*)^n(-\partial_{\gamma^*}+\gamma)^n\}\exp(-|\gamma|^2)P_n(\gamma^*, \gamma) \tag{2.8}$$

where $\overleftarrow{T}$ is the ordering operator which places the derivative operators to the left of the multiplicative operators. Now if we let $\mathscr{Y}_n(\gamma^*, \gamma) = P_n(\gamma^*, \gamma)\exp(-|\gamma|^2)$, $\mathscr{Y}_n(\gamma^*, \gamma)$ is a solution of the differential equation

$$\overleftarrow{T}\{(\partial_\gamma+\gamma^*)^n(-\partial_{\gamma^*}+\gamma)^n\}\mathscr{Y}_n(\gamma^*, \gamma) = \mathscr{Y}_0(\gamma^*, \gamma). \tag{2.9}$$

3.2. Evolution equations in the P representation

Several equations in quantum physics are of the form

$$\mu\partial_x O(x) = [H(x), O(x)] \tag{2.10}$$

where μ is a constant, $H(x)$ and $O(x)$ are two operators depending on the continuous parameter x; $[X, Y] = XY - YX$ is the commutator of X and Y.

To solve (2.10), one usually chooses a complete basis of vectors, with which to calculate matrix elements which are numbers. However, (2.10) has then a complicated structure in terms of the matrix elements of $H(x)$ and $O(x)$ because of their product.

We generalize here some results valid in the coherent state basis to the SCS. Then equations (2.10) are of classical form.

Let us write (2.10) in the 'diagonal' representation:

$$\mu\int d^2\gamma\, \partial_x O_n^{\mathrm{D}}(\gamma^*, \gamma; x)|\gamma, n)(\gamma, n|$$
$$= \int\int d^2\alpha\, d^2\beta(\alpha, n|\beta, n)\{H_n^{\mathrm{D}}(\alpha^*, \alpha; x)O_n^{\mathrm{D}}(\beta^*, \beta; x)$$
$$-O_n^{\mathrm{D}}(\alpha^*, \alpha; x)H_n^{\mathrm{D}}(\beta^*, \beta; x)\}|\alpha, n)(\beta, n|. \tag{2.11}$$

Using (2.2) we obtain:

$$\int d^2\gamma\left(\mu\,\partial_x O_n^{\mathrm{D}}(\gamma; x) - \int\int d^2\alpha\, d^2\beta\{H_n^{\mathrm{D}}(\alpha)O_n^{\mathrm{D}}(\beta) - H_n^{\mathrm{D}}(\beta)O_n^{\mathrm{D}}(\alpha)\}(\alpha, n|\beta, n)K_n(\alpha, \beta, \gamma)\right)$$
$$\times|\gamma, n)(\gamma, n| \equiv 0. \tag{2.12}$$

It is sufficient to have

$$\mu\,\partial_x O_n^{\mathrm{D}}(\gamma^*,\gamma,x) = \int \mathrm{d}^2\alpha\,\mathrm{d}^2\beta H_n^{\mathrm{D}}(\alpha^*,\alpha,x)\{(\alpha,n|\beta,n)K_n(\alpha^*,\alpha,\beta^*,\beta,\gamma^*,\gamma)$$
$$-(\beta,n|\alpha,n)K_n(\beta^*,\beta,\alpha^*,\alpha,\gamma^*,\gamma)\}O_n^{\mathrm{D}}(\beta^*,\beta,x). \tag{2.13}$$

We can let

$$\mathscr{L}_n(\beta,\gamma;x) = \int \mathrm{d}^2\alpha H_n^{\mathrm{D}}(\alpha,x)\{(\alpha|\beta)K_n(\alpha,\beta,\gamma)-(\beta|\alpha)K_n(\beta,\alpha,\gamma)\} \tag{2.14}$$

to obtain

$$\mu\,\partial_x O_n^{\mathrm{D}}(\gamma^*,\gamma,x) = \int \mathrm{d}^2\beta\mathscr{L}_n(\beta^*,\beta,\gamma^*,\gamma;x)O_n^{\mathrm{D}}(\beta^*,\beta;x). \tag{2.15}$$

This equation is very similar to a classical equation of evolution.

3.3. *Evolution equation in the particular case $n = 0$*

In this case, we can produce a more direct derivation. We recall the following result of Ruggeri (1971).

If $B_0^{\mathrm{D}}(\gamma^*,\gamma)$ and $A_0^{\mathrm{D}}(\gamma^*,\gamma)$ are the zeroth order diagonal representations for two operators A and B, then when AB has a diagonal representation of zeroth order, it is:

$$[AB]_0^{\mathrm{D}}(\gamma^*,\gamma) = B_0^{\mathrm{D}}(\gamma^*,\gamma)\exp(-\overleftarrow{\partial}_\gamma\overrightarrow{\partial}_{\gamma^*})A_0^{\mathrm{D}}(\gamma^*,\gamma) \tag{2.16}$$

where the direction of the arrows shows the side where the operator acts.

3.3.1. Diagonal representation for operators: Lemma. Let $F(a^\dagger,a)$ be any bounded operator then $[\overrightarrow{T}F(\gamma^*-\overrightarrow{\partial}_\gamma,\gamma)]\,.\,1$ is its zeroth order diagonal representation.
Proof. If $\overrightarrow{T}$ (or $\overleftarrow{T}$) is the ordering operator which places the derivatives at the right (or the left) of the variables we can write:

$$[AB]_0^{\mathrm{D}}(\gamma^*,\gamma) = [\overrightarrow{T}\{B_0^{\mathrm{D}}(\gamma^*,\gamma-\overrightarrow{\partial}_{\gamma^*})\}]A_0^{\mathrm{D}}(\gamma^*,\gamma) = B_0^{\mathrm{D}}(\gamma^*,\gamma)[A_0^{\mathrm{D}}(\gamma^*-\overleftarrow{\partial}_\gamma,\gamma)\overleftarrow{T}]. \tag{2.18}$$

Since the diagonal representation for $a^\dagger$ (or a) is γ^* (or γ), then:

$$[F(a^\dagger,a)]_0^{\mathrm{D}}(\gamma^*,\gamma) = [\overrightarrow{T}F(\gamma^*,\gamma-\overrightarrow{\partial}_{\gamma^*})]\,.\,1. \tag{2.19}$$

One can also use the derivation of Mehta (1967).

3.3.2. The equation of evolution. According to equation (2.18) we can write equation (2.10) as:

$$\begin{aligned}\mu\,\partial_x O_0^{\mathrm{D}}&(\gamma^*,\gamma)\\ &= [HO]_0^{\mathrm{D}}(\gamma^*,\gamma)-[OH]_0^{\mathrm{D}}(\gamma^*,\gamma)\\ &= [\overrightarrow{T}H_0^{\mathrm{D}}(\gamma^*,\gamma-\overrightarrow{\partial}_{\gamma^*})]O_0^{\mathrm{D}}(\gamma^*,\gamma)-O_0^{\mathrm{D}}(\gamma^*,\gamma)[H_0^{\mathrm{D}}(\gamma^*-\overleftarrow{\partial}_\gamma,\gamma)\overleftarrow{T}].\end{aligned}$$

Then

$$\mu\,\partial_x O_0^{\mathrm{D}}(\gamma^*,\gamma;x) = [\overrightarrow{T}\{H_0^{\mathrm{D}}(\gamma^*,\gamma-\overrightarrow{\partial}_{\gamma^*};x)-H_0^{\mathrm{D}}(\gamma^*-\overrightarrow{\partial}_\gamma,\gamma;x)\}]O_0^{\mathrm{D}}(\gamma^*,\gamma;x). \tag{2.20}$$

This result gives a straightforward method for writing the equation of evolution in the case $n = 0$; it is simpler than the result of Crosignani *et al* (1971).

4. Conclusion

We have proved that the set $\{\exp(\alpha a^{\dagger} - \alpha^{*} a)|n\rangle = |\alpha, n\rangle, n$ given, α complex$\}$ is complete. Using the closure relation thus obtained, we have shown that it is possible to give a diagonal' representation of operators. This remarkable property which generalizes that of Glauber's coherent states has then been used to give an expression for general evolution equations of quantum mechanics.

However, the more complicated analytic expression for the equivalent operators does not seem, up to now, to be very useful, particularly in areas where coherent states give a suitable description.

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